Math 10B with Professor Stankova
Worksheet, Midterm #2; Wednesday, 3/21/2018
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1 Problems

1.1 Bayes Theorem

1. Suppose a test is 99% accurate and 1% of people have a disease. What is the probability that you have the disease given that you tested positive?

Solution: Let B be the event of testing positive and A be the event of having the disease. We want to figure out $P(A|B)$. We know $P(B|A) = 0.99$ and this tells us that we should use Bayes Theorem. Then, we also know $P(B|\bar{A}) = 0.01$ and $P(A) = 0.01, P(\bar{A}) = 0.99$. We can plug this into the formula to get

$$P(A|B) = \frac{1}{1 + \frac{P(B|\bar{A})P(\bar{A})}{P(B|A)P(A)}} = \frac{1}{1 + \frac{0.01 \cdot 0.99}{0.99 \cdot 0.01}} = \frac{1}{2}.$$

2. Suppose I have 3 boxes with 10 red and blue balls in each but with different distributions. Box 1 has 2 red and 8 blue, box 2 has 5 red and 5 blue, and box 3 has 8 red and 2 blue. I randomly pick a box without looking and then pick out a ball. What is the probability that I selected box 3 given that I drew out a red ball?

Solution: Let R be the event that I draw out a red ball and let B_1, B_2, B_3 be the events that I chose boxes $1, 2, 3$ respectively. We want to find $P(B_1|R)$. We know $P(R|B_1)$ because we were told the ball distributions in each box and this tells us that we should use Bayes Theorem. Thus, we have that

$$\begin{aligned} P(B_1|R) &= \frac{P(R|B_1)P(B_1)}{P(R|B_1)P(B_1) + P(R|B_2)P(B_2) + P(R|B_3)P(B_3)} \\ &= \frac{8/10 \cdot 1/3}{8/10 \cdot 1/3 + 5/10 \cdot 1/3 + 2/10 \cdot 1/3} = \frac{8/10}{15/10} = \frac{8}{15}. \end{aligned}$$

3. Suppose I have 3 die. One is 4 sided with numbers $1 - 4$, one is 6 sided with numbers $1 - 6$, and one is 10 sided with sides $1 - 10$. I randomly pick a die without looking and then roll it. What is the probability that I selected the 4 sided die given that I rolled a 1?

Solution: Let A be the event that I roll a 1 and let D_1, D_2, D_3 be the events that I chose die $1, 2, 3$ respectively. We want to find $P(D_1|A)$. We know $P(A|D_1)$ because we were told the sides of each die and this tells us that we should use Bayes Theorem. Thus, we have that

$$P(D_1|A) = \frac{P(A|D_1)P(D_1)}{P(A|D_1)P(D_1) + P(A|D_2)P(D_2) + P(A|D_3)P(D_3)}$$

$$= \frac{1/4 \cdot 1/3}{1/4 \cdot 1/3 + 1/6 \cdot 1/3 + 1/10 \cdot 1/3} = \frac{1/4}{1/4 + 1/6 + 1/10} = \frac{15}{15 + 10 + 6} = \frac{15}{31}.$$

1.2 Random Variables

4. Suppose that we roll two die and let X be equal to the maximum of the two rolls. Find $P(X \in \{1, 3, 5\})$ and draw the PMF for X.

Solution: First we draw the PMF. We calculate $P(X = x)$ by counting the number of ways we can roll two die so that the maximum is x and then dividing by the total number of possibilities, which is 36. So for instance, the only way to get $X = 1$ is if we roll $(1, 1)$ and hence $P(X = 1) = \frac{1}{36}$. Then $P(X = 2) = \{(1, 2), (2, 2), (2, 1)\}/36 = \frac{3}{36}$. Thus, we have that

$$f(1) = \frac{1}{36}, f(2) = \frac{3}{36}, f(3) = \frac{5}{36}, f(4) = \frac{7}{36}, f(5) = \frac{9}{36}, f(6) = \frac{11}{36}.$$

We draw the PMF with stalks at 1 through 6 of those respective heights. Then $P(X \in \{1, 3, 5\}) = P(X = 1) + P(X = 3) + P(X = 5) = \frac{1}{36} + \frac{5}{36} + \frac{9}{36} = \frac{15}{36} = \frac{5}{12}$.

5. I draw 5 cards from a deck of cards. Let X be the number of hearts I draw. What is the range of X and draw the PMF of X. Use this to find the probability that I draw at least 2 hearts.

Solution: The range is $\{0, 1, 2, 3, 4, 5\}$. To calculate $f(x) = P(X = x)$, we count the number of good ways over the total number of ways. The number of good ways to draw x hearts is to first pick out x hearts out of the 13 hearts, and then fill out the rest of the hand and pick $5 - x$ non-heart cards from the remaining 39 cards. Thus $f(x) = \frac{\binom{13}{x}\binom{39}{5-x}}{\binom{52}{5}}$. Thus, we have that $P(X \geq 2) = P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) = \frac{\binom{13}{2}\binom{39}{3} + \binom{13}{3}\binom{39}{2} + \binom{13}{4}\binom{39}{1} + \binom{13}{5}\binom{39}{0}}{\binom{52}{5}}$.

1.3 Discrete Distributions

6. I am picking cards out of a deck. What is the probability that I pull out 2 kings out of 8 cards if I pull with replacement? What about without replacement?

> **Solution:** With replacement is repeated Bernoulli trials which means binomial distribution. The probability of a success or pulling out a heart is $\frac{1}{13}$. Therefore, the probability of pulling 2 kings out of 8 is
>
> $$\binom{8}{2}\left(\frac{1}{13}\right)^2 \cdot \left(\frac{12}{13}\right)^{8-2}.$$
>
> If do not have replacement, then this is a hyper-geometric distribution with $N = 52, n = 8, m = 4$, the the answer is
>
> $$\frac{\binom{4}{2}\binom{48}{6}}{\binom{52}{8}}.$$

7. What is the probability that first king is the third card I draw (with replacement)?

> **Solution:** We want to know the probability of the first success, which is geometric since we are doing with replacement. The probability of a success is $p = \frac{1}{13}$ and we have two failures before we have a success so $k = 2$. Hence the answer is $f(2) = (12/13)^2(1/13)$.

8. In a class of 40 males and 60 females, I give out 4 awards randomly. What is the probability that females will win all 4 awards if the awards must go to different people? What about if the same person can win multiple awards?

> **Solution:** This is like the probability of picking 4 females out of 4 people chosen. If the awards must go to different people, there is no replacement so it is the hypergeometric distribution where a success is picking a female. So we have $N = 100$ students total, there are $m = 60$ females, and I am picking $n = 4$ students and I want $k = 4$ females. This gives
>
> $$f(4) = \frac{\binom{60}{4}\binom{40}{0}}{\binom{100}{4}} = \frac{\binom{60}{4}}{\binom{100}{4}}.$$
>
> If the same person can will all the awards, then we are choosing with replacement. So, this is a binomial distribution where the probability of success is $p = \frac{60}{100}$. Thus,

we have that the answer is

$$f(4) = \binom{4}{4}\left(\frac{60}{100}\right)^4\left(\frac{40}{100}\right)^0 = \frac{3^4}{5^4}.$$

9. In a class of 50 males and 90 females, I give out 4 awards randomly. What is the probability that females will win 2 awards if the awards must go to different people? What about if the same person can win multiple awards?

Solution: This is like the probability of picking 2 females out of 4 people chosen. If the awards must go to different people, there is no replacement so it is the hypergeometric distribution where a success is picking a female. So we have $N = 140$ students total, there are $m = 90$ females, and I am picking $n = 4$ students and I want $k = 2$ females. This gives

$$f(2) = \frac{\binom{50}{2}\binom{90}{2}}{\binom{140}{4}}.$$

If the same person can will all the awards, then we are choosing with replacement. So, this is a binomial distribution where the probability of success is $p = \frac{90}{140}$. Thus, we have that the answer is

$$f(2) = \binom{4}{2}\left(\frac{90}{140}\right)^2\left(\frac{50}{140}\right)^2.$$

10. At Berkeley, 3/4 of the population is undergraduates. I cold call someone at random and ask for their age. What is the probability that I have to call 10 people before I call an undergraduate? What is the probability that I call 4 undergraduates out of 10 people I call (if I can call someone more than once)?

Solution: The probability of calling an undergrad is $\frac{3}{4}$. The probability that I call 10 people before the first undergrad is given by the geometric distribution since we are talking about times until a success. I have 10 failures before and so this is

$$f(10) = (1 - 3/4)^{10}(3/4) = \frac{3}{4^{11}}.$$

The probability that I call 4 undergraduates out of 10 people is given by a binomial distribution since I can call someone more than once and hence there is replacement.

So plugging this into the binomial distribution gives

$$f(4) = \binom{10}{4}(3/4)^4(1/4)^6.$$

11. In a dorm of 100 people, there are 20 people who are underage. I go to a party with 40 people. What is the probability that there is at least one underage person there?

Solution: The problem says at least which clues te fact that we should think about complimentary probability. The probability that there is at least one underage person is 1 minus the probability that there are no underage people there. In order to calculate this latter probability, we determine what kind of distribution this is. There are $m = 20$ “tagged” people who are underage and out of a total population of $N = 100$ people, we want to choose $n = 40$ people and want to get $k = 0$ minors. This is a hyper-geometric distribution because we are picking people without replacement and hence the probability of having at least one underage person is

$$1 - \frac{\binom{20}{0}\binom{80}{40}}{\binom{100}{40}} = 1 - \frac{\binom{80}{40}}{\binom{100}{40}}.$$

12. I roll a fair 6-sided die over and over again until I roll a 6. What is the probability that it takes me more than 10 tries?

Solution: Let X be the number of failures before a success (rolling a 6). Then X is given by a geometric distribution with $p = \frac{1}{6}$. We want to know $P(X \geq 10)$ which we can calculate as

$$P(X \geq 10) = P(X = 10) + P(X = 11) + P(X = 12) + \cdots = (1-p)^{10}p + (1-p)^{11}p + \cdots.$$

This is a geometric series which is equal to $\frac{(1-p)^{10}p}{1-(1-p)} = (1-p)^{10}$. In fact, for a geometric series, we have $P(X \geq n) = (1-p)^n$ and look at Homework 15 for the justification.

13. The number of chocolate chips in a cookie is Poisson distributed with an average of 15 chocolate chips. What is the probability that you pick up a cookie with only 10 chocolate chips in it?

Solution: This is a Poisson distribution with $\lambda = 15$. We want to calculate $f(10) = \frac{\lambda^{10}e^{-\lambda}}{10!} = \frac{15^{10}e^{-15}}{10!}$.

14. The number of errors on a page is Poisson distributed with approximately 1 error per 50 pages of a book. What is the probability that a novel of 300 pages contains at most 1 error?

Solution: If we average 1 error per 50 pages, then over a novel of 300 pages, we should expect $300/50 \cdot 1 = 6$ errors. Thus, the probability of having at most one error with $\lambda = 6$ is $f(0) + f(1) = \frac{\lambda^0 e^{-\lambda}}{0!} + \frac{\lambda^1 e^{-\lambda}}{1!} = e^{-6} + 6e^{-6} = 7e^{-6}$.

15. I roll two fair 6 sided die. What is the expected value of their product?

Solution: Let X be the first value I roll and Y be the second. The rolls are independent and so $E[XY] = E[X]E[Y] = 3.5 \cdot 3.5 = 12.25$.

16. While pulling out of a box of cookies, what is the expected number of cookies I have to pull out before I pull out an oatmeal raisin if 25% of cookies are oatmeal raisin and I choose with replacement? What is the variance?

Solution: This is a geometric distribution because I am counting the number of cookies I have to pull out before a success. The probability of success is $25\% = p = 1/4$. So the expected number of cookies I have to pull out is $\frac{1-p}{p} + 1 = 3 + 1 = 4$. The variance is $\frac{1-p}{p^2} = 3/(1/4) = 12$.

17. What is the expected number of aces I have when I draw 5 cards out of a deck?

Solution: Drawing cards out of a deck without replacement is the hypergeometric distribution. There are $N = 52$ cards total and $m = 4$ aces total. Then, we pull out $n = 5$ cards and so the expected number of aces is $\frac{mn}{N} = \frac{20}{52}$.

18. In a safari, safari-keepers have caught and tagged 300 rhinos. On a safari, out of the 15 different rhinos you see, there are 5 of them expected to be tagged. How many rhinos are there at the safari?

Solution: This is a hyper-geometric distribution because out of the N rhinos total and $m = 300$ tagged rhinos, you see that $n = 15$ rhinos that you see, there are 5 of them expected to be tagged. So $5 = E(X) = \frac{mn}{N} = \frac{300 \cdot 15}{N}$. So $N = \frac{300 \cdot 15}{5} = 900$.

1.4 Hypothesis Testing

19. Chip bags say that they have 14 ounces of chips inside with a standard deviation of 0.5 ounces. You weigh 100 bags and get an average of 13.8 ounces. Can you say that they are underselling you with significance level $\alpha = 0.05$?

Solution: The null hypothesis is that the chips have a weight of 14 ounces. The central limit theorem tells us that the standard deviation of the sample is $0.5/\sqrt{n} = 0.5/\sqrt{100} = 0.05$. We calculate the p value is $1/2 - z(\frac{|a-\mu|}{\sigma/\sqrt{n}}) = 1/2 - z(\frac{|13.8-14|}{0.05}) = 1/2 - z(4) < \alpha$. Thus, we can say that the chip bag makers are lying.

20. You flip a coin 100 times and get 60 heads. Can you say the coin isn't fair with significance level $\alpha = 0.05$?

Solution: We need to first assume the null hypothesis. This is that the coin is fair and $p = 1/2$. Now when we flip the coin 100 times, the central limit theorem tells us that the percentage of heads that we get will be p as before and standard deviation is $\sqrt{p(1-p)}/\sqrt{n} = 1/2/\sqrt{100} = 0.05$. So the probability of getting at least an extreme scenario as 60 heads is $\frac{1}{2} - z(|0.6 - 0.5|/0.05) = \frac{1}{2} - z(2) = 0.5 - 0.4772 = 0.0228 < \alpha$. Thus we can reject the null hypothesis and say that the coin is biased towards heads.

21. You roll a 10 sided die 100 times and get 5 16 times. Can you say the die is biased towards 5 with significance level $\alpha = 0.05$?

Solution: We need to first assume the null hypothesis. This is that the die is fair and $p = 1/10$. Now when we roll the die 100 times, the central limit theorem tells us that the percentage of 5s we get is approximately normally distributed with mean $p = \frac{1}{10}$ and standard deviation $\sqrt{p(1-p)}/\sqrt{n} = \frac{3}{100}$. So the probability of getting at least an extreme scenario as 16 5s is $\frac{1}{2} - z(|0.16 - 0.10|/0.03) = \frac{1}{2} - z(2) = 0.5 - 0.4772 = 0.0228 < \alpha$. Thus we can reject the null hypothesis and say that the coin is biased towards heads.

22. You take 400 cards and get 107 spades, 112 hearts, 75 diamonds, and 106 clubs. Can you say that the suits are not evenly distributed with $\alpha = 0.05$?

Solution: If they are evenly distributed, we would expect 100 of each. Calculating the statistic gives

$$r = \frac{(107-100)^2}{100} + \frac{(112-100)^2}{100} + \frac{(75-100)^2}{100} + \frac{(106-100)^2}{100}$$
$$= \frac{49+144+625+36}{100} = 8.54.$$

For $4-1=3$ degrees of freedom, our critical value for α is 7.815 and since $8.54 > 7.815$, we reject the null hypothesis and say that the cards are not uniformly distributed.

23. In a M&M bag, you get 12 brown ones, 13 yellow ones, 12 red, 14 green, 23 blue, and 22 orange. Is it possible that the colors are evenly distributed with a significance level of $\alpha = 0.05$?

Solution: In 96 MMs, we expect to get 16 of each. Following the formula, our statistic is:

$$\frac{(12-16)^2}{16} + \frac{(13-16)^2}{16} + \frac{(12-16)^2}{16} + \frac{(14-16)^2}{16} + \frac{(23-16)^2}{16} + \frac{(22-16)^2}{16}$$
$$= \frac{16+9+16+4+49+36}{16} = 8.125.$$

There are 6 options so we have $6-1=5$ degrees of freedom. For 5 degrees of freedom and $\alpha = 0.05$, our critical value is 11.071. Since $8.125 < 11.071$, we fail to reject the null hypothesis that the colors are evenly distributed.

24. You want to know whether living location is independent of a students grade. You interview some people and get the following results:

	Sophomore	Junior	Senior
South	20	30	50
Downtown	10	30	60
North	10	40	50

Are they independent?

Solution: We fill out the table with the sums as:

	Sophomore	Junior	Senior	
South	20	30	50	100
Downtown	10	30	60	100
North	10	40	50	100
	40	100	160	300

Now we can fill out the expected value table. For instance, the expected number of juniors living downtown is $\frac{100}{300} \cdot \frac{100}{300} \cdot 300 = \frac{100}{3}$. Filling this out gives:

	Sophomore	Junior	Senior	
South	40/3	100/3	160/3	100
Downtown	40/3	100/3	160/3	100
North	40/3	100/3	160/3	100
	40	100	160	300

Computing the statistic gives 8.25. The critical value for $(3-1)(3-1) = 4$ degrees of freedom is 9.488. Thus, we cannot reject the null hypothesis.

25. You are wondering whether performing well in this course and gender are related and you get the following table. Are they related?

	Male	Female
Pass	315	485
Fail	85	115

Solution: There are a total of 800/1000 people who pass and 200/1000 who fail, and 400/1000 who are male and 600/1000 who are female. Thus, if they were independent, for instance we would expect that $\frac{800}{1000} \cdot \frac{600}{1000} = 48\%$ of people to be female and pass. We can fill out the expected table as follows:

	Male	Female
Pass	320	480
Fail	80	120

Now we can do the χ^2 test to get a statistic of

$$\frac{(315-320)^2}{320} + \frac{(485-480)^2}{480} + \frac{(85-80)^2}{80} + \frac{(115-120)^2}{120} = 0.651.$$

The critical value for 1 degree of freedom is 3.841 and $0.651 < 3.841$ so we cannot reject the null hypothesis.

1.5 Recursion Equations

26. Solve the recurrence relation $a_n = 3a_{n-1} + 2$ with $a_0 = 0$.

Solution: This is a non-homogeneous first order linear equation. So, we have a formula for this. The formula given in class says that the general solution of $a_n = \alpha a_{n-1} + \beta$ is $a_n = \alpha^n a_0 + \beta(\frac{\alpha^n - 1}{\alpha - 1})$. In this case, we have $\alpha = 3$ and $\beta = 2$ so the solution is

$$a_n = 3^n a_0 + 2\frac{3^n - 1}{3 - 1} = 3^n - 1$$

since $a_0 = 0$.

27. Verify that $a_n = \binom{2n}{n}$ is a solution to $a_n = \frac{4n-2}{n} a_{n-1}$.

Solution: We plug in $\binom{2n}{n}$ to get $\binom{2n}{n} \stackrel{?}{=} \frac{4n-2}{n}\binom{2n-2}{n-1} = \frac{4n-2}{n}\frac{(2n-2)!}{(n-1)!(n-1)!} = \frac{2\cdot(2n-1)!}{n!(n-1)!} = \frac{(2n)\cdot(2n-1)!}{n!n(n-1)!} = \binom{2n}{n}$ so this is indeed a solution.

28. Solve the recurrence relation $a_n = 3a_{n-1} + 4a_{n-2}$ with $a_0 = 3$ and $a_1 = 2$.

Solution: The characteristic polynomial is $\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0$. Thus the general form is $a_n = c_1 4^n + c_2(-1)^n$. Plugging in our initial conditions gives $c_1 + c_2 = 3$ and $4c_1 - c_2 = 2$ which gives $c_1 = 1$ and $c_2 = 2$. So the answer is $a_n = 4^n + 2 \cdot (-1)^n$.

29. Solve the recurrence relation $a_n = 5a_{n-1} - 4a_{n-2}$ with $a_0 = 3$ and $a_1 = 6$.

Solution: The characteristic polynomial is $\lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1) = 0$. Thus the general form is $a_n = c_1 4^n + c_2 1^n$. Plugging in our initial conditions gives $c_1 + c_2 = 3$ and $4c_1 + c_2 = 6$ which gives $c_1 = 1$ and $c_2 = 2$. So the answer is $a_n = 4^n + 2$.

30. Find A, B such that $a_n = An + B$ is a solution to the recurrence relation $a_n = 2a_{n-1} + n$.

Solution: We plug it in to get $An + B = 2(A(n-1) + B) + n = 2An - 2A + 2B + n$ and simplifying gives $An + n + B - 2A = 0$. This is true for all n and so we can just plug in values of n to solve. First try $n = 0$ to get $B - 2A = 0$ or $B = 2A$. Now plug in $n = 1$ to get $A + 1 + B - 2A = B - A + 1 = 0$. So $B = A - 1 = 2A$ so $A = -1$ and $B = 2A = -2$. Thus the solution is $-n - 2$.

One good thing to do now is plug this back in and make sure that this works. We get

$$-n - 2 \stackrel{?}{=} 2(-(n-1) - 2) + n = -2n + 2 - 4 + n = -n - 2.$$

Thus, this indeed is a solution.

1.6 Differential Equations

31. Find a solution to $xy' + y = 3x^2$ with $y(1) = 5$.

Solution: First, we need to make sure the coefficient of y' is 1 and we do this by dividing by x. So, we have an equation $y' + \frac{y}{x} = 3x$. Now our $P(x) = 1/x$ and the integrating function that we multiply by is $I(x) = e^{\int P(x)dx} = e^{\int 1/x dx} = e^{\ln x} = x$. Multiplying through gives us the same thing again $xy'+y = 3x^2$ but now we recognize on the left side is $(I(x)y)' = (xy)'$. So, we have that $(xy)' = 3x^2$ and we can integrate both sides to get $xy = x^3 + C$ or $y = x^2 + \frac{C}{x}$. Finally at this point, we plug in our initial condition that $y(1) = 5$ or that $5 = 1 + \frac{C}{1} = 1 + C$ so $C = 4$. So, our solution is $y = x^2 + \frac{4}{x}$.

32. Find the solutions to $y' = y\tan(x) - \sec(x)$.

Solution: First we bring the ys on one side to get $y' - \tan(x)y = -\sec(x)$. Our integrating factor is then $e^{\int -\tan(x)dx} = e^{\ln\cos(x)} = \cos(x)$. Multiplying by this, we get $\cos(x)y' - \tan(x)\cos(x)y = (\cos(x)y)' = -\sec(x)\cos(x) = -1$. Integrating gives $\cos(x)y = -x + C$ so $y = \frac{C-x}{\cos(x)}$ is the general solution.

33. Find the solutions to $e^x y' + y = 1$ with $y(0) = 1$.

Solution: First get the coefficient of y' to be 1 by dividing by e^x. This gives $y' + e^{-x}y = e^{-x}$. So, our integrating factor is $e^{\int e^{-x}dx} = e^{-e^{-x}}$ and we get that $e^{-e^{-x}}y' + e^{-x}e^{-e^{-x}}y = (e^{-e^{-x}}y)' = e^{-x}e^{-e^{-x}}$. Integrating, we let $u = e^{-x}$ and hence $\int e^{-x}e^{-e^{-x}}dx = \int(-du)e^{-u} = e^{-u} = e^{-e^{-x}}$. Thus $e^{-e^{-x}}y = e^{-e^{-x}} + C$ and so the solution is $y = 1 + Ce^{e^{-x}}$. Plugging in the initial condition that $y(0) = 1$, we get $1 = 1 + Ce^1 = 1 + Ce$ so $C = 0$. Thus, the solution is just $y = 1$.

2 True/False

34. True **FALSE** To find $P(B|A)$, it suffices to know just $P(A|B)$ and how to apply Bayes' Theorem.

Solution: We also need to know $P(B)$ and $P(A|\bar{B})$.

35. **TRUE** False Among other things, the proof of Bayes' Theorem for finding $p(B|A)$ depends on being able to split the probability $p(A)$ as a sum probabilities $p(A \cap B)$ and $p(A \cap \overline{B})$, and then further rewrite these as products of certain other probabilities.

36. True **FALSE** The extra shortcut formula $p(B|A) = \frac{1}{1+\frac{p(A|\overline{B})\cdot p(\overline{B})}{p(A|B)\cdot p(B)}}$ works in one particular case when the standard formula for $p(B|A)$ in Bayes' Theorem fails.

> **Solution:** It is the opposite. The standard formula works in a case when this shortcut does not, which is when $P(A \cap B) = 0$.

37. True **FALSE** If a winner in a bicycle race tests positive for steroids, and this test has a very high "True Positive" rate and hence a very low "False positive" rate, then we should take away the winning cup from the athlete because it is extremely likely that he/she has used steroids.

> **Solution:** The actual answer could be a lot different from just the true positive rate.

38. **TRUE** False Error 1 in Hypothesis Testing (reject the null-hypothesis that the person is healthy when the person is actually healthy) is analogous to Testing positive for steroids (event T), yet not having used steroids (event $\overline{S}$); in other words, the significance α corresponds to $p(T \cap \overline{S})$.

39. **TRUE** False Error 2 in Hypothesis Testing (keep the null-hypothesis that the person is healthy but the person is, in fact, sick) is analogous to Testing negative for steroids (event $\overline{T}$), yet having used steroids (event S); in other words, the power of a test $1 - \beta$ corresponds to $1 - p(\overline{T} \cap S)$.

40. True **FALSE** To partition a set Ω into a disjoint union of subsets $B_1, B_2, \ldots, B_n$, means that the intersection of these sets is empty; i.e., $B_1 \cap B_2 \cap \cdots \cap B_n = \emptyset$.

> **Solution:** We need the pairwise intersections $B_i \cap B_j$ to be empty as well.

41. True **FALSE** Two disjoint events could be independent, but two independent events can never be disjoint.

Solution: If one event is the empty set, then it is disjoint and independent with any other event.

42. True **FALSE** If a fair coin comes up Heads six times in a row, it is more likely that it will come up Tails than Heads on the 7th flip.

Solution: If it is fair, the flips are independent.

43. **TRUE** False Contrary to how we may use the word "dependent" in everyday life; e.g., event A could be dependent on event B, yet event B may not be dependent on event A; in math "dependent" is a symmetric relation; i.e., A is dependent with B if and only B is dependent with A.

44. **TRUE** False If A and B are independent events, their complements are also bound to be independent, and to prove this we need a general argument since an example is not sufficient here.

45. True **FALSE** If A and B are independent events, $\overline{A}$ and B may fail to be independent, but to prove this we need just one counterexample, not a general proof.

Solution: They are also independent.

46. True **FALSE** If any pair of events among $A_1, A_2, ..., A_n$ are independent, then all events are independent.

47. True **FALSE** A random variable (RV) on a probability space (Ω, P) is a function $X : \Omega \to \mathbb{R}$ that satisfies certain rules and is related to the probability function P.

Solution: A RV is **any** function to $\mathbb{R}$ and not at all related to the probability function P. The PMF of this RV is related to P though.

48. **TRUE** False A RV X could be the only source of data for an outcome space Ω and hence could be very useful in understanding better X 's domain.

Solution: One reason we introduced random variables is because it is sometimes hard to understand all of Ω and we can only look at it through X.

49. True **FALSE** The notation "$X \in E$" means that the RV X starts from event $E \subseteq \Omega$ and lands in $\mathbb{R}$.

Solution: The notation is for $E \subset \mathbb{R}$ and is used to denote the event the RV X lands in E.

50. True **FALSE** The notation "$X^{-1}(E)$" for a RV X and event $E \subseteq \Omega$ means to take set $B \subseteq \Omega$ of the reciprocals of all elements in E that are in the range of X .

Solution: The notation $X^{-1}(E) \subset \Omega$ is for $E \subset \mathbb{R}$ and denotes all outcomes that are in the preimage of E.

51. True **FALSE** The PMF of a RV X on probability space (Ω, P) is a third function $f : \mathbb{R} \to [0, 1]$ such that the composition of X followed by f on any $\omega \in \Omega$ is equal to P; i.e., such that $f(X(\omega)) = P(\omega)$.

Solution: First off, the probability function takes in subsets of Ω and hence $P(\omega)$ does not make sense. Even if we were to replace it with $P(\{\omega\})$, this would still potentially be wrong as seen in TF Question 9. If $x = X(\omega)$, by definition, we have $f(X(\omega)) = P(X^{-1}(x))$ and $X^{-1}(x)$ could contain more things than just ω.

52. True **FALSE** It is possible that $f(x) > P(x)$ for some $\omega \in \Omega$ and the corresponding $x = X(\omega) \in \mathbb{R}$ where X a discrete RV on (Ω, P) with PMF f .

Solution: If we replace $P(x)$ with $P(\{\omega\})$, then the result is true (note that $x \in \mathbb{R}$ and so $P(x)$ doesn't make sense because both $x \notin \Omega$ and it is a subset). We have $f(X(\omega)) = P(X = X(\omega))$ so the probability of all outcomes that are mapped to the same value as ω. For example, if we are counting the number of boys in a family, then $f(X(BBG)) = P(\{BBG, BGB, GBB\}) > P(\{BBG\})$.

53. True **FALSE** To show that two RV's $X, Y : \Omega \to \mathbb{R}$ are independent on (Ω, P) , we can find two subsets $E, F \subseteq \mathbb{R}$ for which $P(X \in E \text{ and } Y \in F) = P(X \in E) \cdot P(Y \in F)$.

Solution: To show that they are independent, we need to show that for all subset E, F, the equality holds. To show that they are not independent, we only need to find a single counterexample.

54. True **FALSE** Two Bernoulli trials are independent only if the probability of success and failure are each $\frac{1}{2}$.

Solution: Bernoulli trials can be independent regardless of what p is.

55. **TRUE** False The product X and the sum Y of the values of two flips of a fair coin (H=1, T=0) are dependent random variables.

Solution: If we know that $Y = 2$, then we know we got 2 heads and hence we know that $X = 1$, showing that they aren't independent.

56. **TRUE** False To turn the experiment of "rolling a die once" into a Bernoulli trial, we need to split its outcome space into two disjoint subsets and declare one of them a success.

57. **TRUE** False Several Bernoulli trials performed on one element at a time from a large outcome space Ω, without replacement, are approximately independent because what happens in one Bernoulli trial hardly affects the ratio of "successes" to "failures" in the remainder of the population.

58. True **FALSE** The probability of having 20 women within randomly selected 40 people is about 50%, assuming that there is an equal number of women and men on Earth.

Solution: This is a Poisson distribution with average number of women $\lambda = 20$. But the probability of getting exactly 20 women is $f(20) = \frac{20^{20}e^{-20}}{20!} \approx 9\%$.

59. **TRUE** False The hypergeometric distribution describes the probability of k "successes" in n random draws without replacement from a population of size N that contains exactly m "successful" objects.

60. True **FALSE** While the hypergeometric and binomial probabilities depend each on 3 parameters and 1 (input) variable, the Poisson probability depends only on 1 parameter and 1 (input) variable.

Solution: The binomial distribution depends on two variables (n and p)

61. True **FALSE** To approximate well the probability of k successes in a large number n of independent Bernoulli trials, each with individual probability of success p that is relatively small, we can use the formula $\frac{(np)^k e^{-np}}{n!}$.

Solution: The denominator should have $k!$ instead of $n!$.

62. **TRUE** False $E(X-Y) = E(X) - E(Y)$ for any R.V.s X and Y, regardless of whether they are independent or not.

63. **TRUE** False $Var(X) = E(X^2) - E^2(X)$ holds true because, essentially, the expected value has linearity properties.

Solution: By definition $Var(X) = E[(X - E[X])^2]$ and then we expand by using linearity to get the above result.

64. **TRUE** False Splitting a R.V. X as a sum of simpler R.V.'s X_i could be advantageous when we want to compute $Var(X)$, but we need to be careful that these X_i's are independent.

65. **TRUE** False If X is the geometric R.V., then $Var(X) = \frac{1-p}{p^2}$.

66. **TRUE** False If X is the hypergeometric R.V. in variable k and with parameters m, n, N, then $N = \frac{mn}{E(X)}$.

67. **TRUE** False If Santa Claus randomly throws n presents into n chimneys (one present per chimney), on the average one home will receive their intended present, regardless of how large or small n is.

Solution: This is similar to the hat example.

68. **TRUE** False For any independent R.V's X and Y, we have $Var(aX + bY) = a^2 Var(X) + b^2 Var(Y)$, but $Var(XY) = Var(X) \cdot Var(Y)$ only if $E(X) = 0 = E(Y)$ or $X = 0$ or $Y = 0$ or both X and Y are constants.

Solution: This comes from studying the equation $Var(XY) = Var(X)Var(Y) + E[X]^2Var(Y) + E[Y]^2Var(X)$ and asking when can $E[X]^2Var(Y) = E[Y]^2Var(X) = 0$.

69. **TRUE** False As long as several RV's $X_1, X_2, ..., X_n$ are identically distributed, their average RV $\overline{X}$ will have the same mean as each of them, but the standard error of $\overline{X}$ may or may not be equal to $\frac{SE(X_3)}{\sqrt{n}}$.

Solution: The mean will always be the same but in order for the standard deviation to be equal to that, we need independence of the variables.

70. True **FALSE** The formula for the mean of the average $\overline{X} = \frac{X_1+X_2+\cdots+X_n}{n}$ of independent identically distributed RV's has $\sqrt{n}$ in the denominator.

Solution: The mean does not have a $\sqrt{n}$ in the denominator but the standard deviation does.

71. True **FALSE** The formula for the variance $Var(X+Y) = Var(X) + Var(Y)$ works regardless of whether the RV's X and Y are independent or not.

72. **TRUE** False To approximate the height of a tall tree (using similar triangles and measurements along the ground – without climbing the tree!) it is better to ask several people to do it independently of each other and then to average their results, than to do it once just by yourself.

Solution: As n gets higher, your 95% confidence interval becomes smaller.

73. True **FALSE** According to the Central Limit Theorem, the normalized distribution $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ is the standard normal distribution for n large, where $\overline{X}$ is the average of n independent, identically distributed variables, each with mean μ and standard error $\sigma/\sqrt{n}$.

Solution: Each of the X_i has standard error σ, not $\sigma/\sqrt{n}$.

74. True **FALSE** If we have data $\{x_1, x_2, \ldots, x_N\}$ from the whole population, we should divide by n instead of $(n-1)$ in the formula for the sample standard deviation.

Solution: We need to divide by $n-1$ to get the sample standard deviation which is the unbiased estimator.

75. True **FALSE** The smaller the 95% confidence interval is, the lower our confidence is that the true parameter is in that interval.

Solution: It represents our 95% confidence range and hence we are always 95% sure it is in the interval even though it may have different sizes in different problems.

76. True **FALSE** To reject the null hypothesis H_0 when it is actually true is equivalent to making a Type 2 error with significance α.

Solution: This is a type 1 error.

77. True **FALSE** The null hypothesis is a theory that we believe is true.

Solution: We often want to show that the null hypothesis is wrong.

78. **TRUE** False The higher the significance of a test, the higher the probability of rejecting a true null hypothesis.

79. True **FALSE** Adding up the power and the significance of a test yields 1.

Solution: The values α and β have no relationship to each other.

80. **TRUE** False A type-2 error made by a road patrol may result in letting drunken drivers continue driving.

81. **TRUE** False The significance of a test shows how often, on the average, we can make a Type 1 error.

82. **TRUE** False Using two-sided Alternative Hypotheses H_1 may lead to twice as large significance as their one-sided analogs.

83. **TRUE** False The p-value of a possible test result r is the probability that the experiment produces a result that is equally or more extreme (towards H_1) than r, assuming H_0 is true.

84. True **FALSE** In class we concluded that marital status and employment status for men in the age group 35-44 years old must be dependent since the resulting r-value of the test statistic $R = \sum_{i,j} \frac{(N_{ij}-M_{ij})^2}{M_{ij}}$ came to $r \approx 31.61 > \alpha = 0.05$.

Solution: We calculated the area and compared that to α and didn't compare the value directly with the probability.